# A Online Appendix

## A.1 Definitions

Here, we first include the technicalities necessary to define equilibrium. We prove results based on the general model and derive the results for the simple model as a special case. We then provide the proof for competitive clientelism.

Voters are characterized by their bias in favor of the party machine, $v$. We interpret this valuation as the benefit the voter would need to receive in order to be made indifferent between voting for the machine and voting for the opposition. If $v > 0$, the voter is biased towards the opposition and requires a payment to vote for the machine. If $v < 0$, then the voter requires a payment to vote for the opposition. If $v = 0$, the voter is precisely indifferent between the parties. We assume that the distribution of $v$ is characterized by the density function $f_v$ and we let $F_v$ denote the cumulative distribution function of the voter valuations.

We can relate this framework to the simple model previously considered using $v = \frac{1}{\gamma}|x - 0| - \frac{1}{\gamma}|x - 1| = \frac{2x-1}{\gamma}$ and $v \sim U[-\frac{1}{\gamma}, \frac{1}{\gamma}]$. The value $v$ represents the policy utility of voting for the party machine minus the policy utility of voting for the opposition, divided by the weight of particularistic benefits. A voter with ideal point 0 would require $-\frac{1}{\gamma}$ pesos to vote for the party machine, meaning she would be indifferent between voting for the machine and receiving $\frac{1}{\gamma}$ pesos and voting for the opposition.

This model is more general than the basic model developed above. For example, it can allow for an arbitrary distribution of $x$, an arbitrary multi-dimensional symmetric utility function, and a valence advantage of one party. In this case, we would have $v = u(||x-x_M||)+z-u(||x-x_O||)$ where $u$ is a strictly decreasing function, $x_M$ and $x_O$ are the positions of the machine and opposition, $||.||$ is a distance metric, and $z$ is valence advantage of the machine party, which may be negative. The model can allow $x$ to have an arbitrary density $f_x$ and can even allow the perceived valence advantage to differ across voters and be correlated with their ideal points by assuming that $(x, z)$ follows the joint density $f_{x,z}$. In this case, the equation $v = u(||x - x_M||) + z - u(||x - x_O||)$ would define the implied distribution for $v$ and the model can be analyzed by focusing on $v$ and $f_v$ without direct reference to the model primitives.

The voter strategies are maps $\lambda : \mathbb{R} \to \mathbb{R}^+$ from the real line to the positive real line. Each $\lambda(v)$ gives the demand a voter characterized by bias $v$ makes to the party machine. We require all strategy profiles to be measurable functions. We thus define the strategy profiles for the players as the set of measurable functions mapping the real line to the positive real line and we denote this space by $\Lambda$.

The party machine moves second. For each history $\lambda \in \Lambda$, the party machine's strategy is a map of the form $b_\lambda : \mathbb{R} \to \mathbb{R}^+$. Each $b_\lambda(v)$ gives the benefit that is proposed to a voter characterized by bias $v$ when the history $\lambda$ is reached. We require the function $b_\lambda$ to be measurable at each history $\lambda \in \Lambda$. For each history $\lambda \in \Lambda$, the party machine strategy must also satisfy the budget constraints $b_\lambda(v) \geq 0$ for all $v \in \mathbb{R}$ and $\int_{v\in\mathbb{R}} b_\lambda(v) f_v(v)dv \leq B$. We thus define the party machine strategies to be the set of measurable functions from the real line to the positive real line satisfying the budget constraints $b_\lambda(v) \geq 0$ for all $v \in \mathbb{R}$ and $\int_{v\in\mathbb{R}} b_\lambda(v) f_v(v)dv \leq B$. We denote this set by $\mathcal{B}_\lambda$ and we denote $\mathcal{B} = \{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$.

We let $(\lambda, b) \in \Lambda \times \mathcal{B}$ denote a strategy profile. We let $\lambda_{-v} = \{\lambda(v')\}_{v'\in\mathbb{R}-v}$ denote the voter strategy profile for all voters except $v$. A voter with bias $v$ has a utility function given by,

$$u_v(b, \lambda) = \begin{cases} -v + b_\lambda(v), & b_\lambda(v) \geq \lambda(v) \\ b_\lambda(v), & b_\lambda(v) < \lambda(v) \end{cases}$$

The party machine has a utility function given by,

$$U(b,\lambda) = U_{sm}\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

where $U_{sm}(s,m)$ satisfies the following assumptions,

**Assumption 1.** *$U_{sm}(s',m) \geq U_{sm}(s,m)$ for $s' \geq s$*

**Assumption 2.** *$U_{sm}(s,m') \geq U_{sm}(s,m)$ for $m' \geq m$*

**Assumption 3.** *$U_{sm}(s',m') > U_{sm}(s,m)$ for $s' > s$ and $m' > m$*

**Assumption 4.** *$U_{sm}(s,m)$ is continuous in $s$ for all $(s,m)$ such that $s \neq \frac{1}{2}$, right-continuous in $s$ when $s = \frac{1}{2}$, and continuous in $m$ for all $(s,m)$*

**Assumption 5.** *Either $U_{sm}(1,m') > U_{sm}(1,m)$ for all $m' > m$ or there exists a $\tau$ with $f_v(\tau) > 0$ such that $\tau F_v(\tau) > B$*

Throughout the appendix, we will use a slight abuse of notation and abbreviate $U_{sm}(s,m)$ as $U(s,m)$. Assumption 1 implies that the party machine's utility is weakly increasing in its' vote share. Assumption 2 implies that the party machine's utility is weakly increasing in its' left-over money. We assume that party machine utility strictly increases if both vote share and unspent money increase (Assumption 3). We also assume that the party machine utility function is continuous everywhere except $s = \frac{1}{2}$ to allow for the party machine to receive a discrete positive benefit from winning the election (Assumption 4). Assumption 5 requires that either the party machine avoids wasting money when it has already bought the entire electorate or that the party machine does not have enough resources to buy the entire electorate.

We highlight a number of special cases that are consistent with the assumptions above. We can assume that the party machine maximizes its' vote share $U(s,m) = s$. This is the case considered in the literature (Stokes 2005; Nichter 2008; Gans-Morse et al. 2014). In the literature, it is also common to assume that the preferences of voters have a uniform distribution. In this case, in order to satisfy assumption 5, we would have to assume that the machine budget is not too large. This is the $B < \gamma^{-1}$ condition assumed in the body of the paper. The case where parties maximize vote share is actually the most technically difficult case to consider, but we include it because otherwise we would not be able to compare our results to existing results in the literature. If we had alternatively assumed that the distribution of $v$ had full support, or even just that it has a right tail, Assumption 5 would have been satisfied automatically.

Alternatively, we could allow $U(s,m) = \beta 1\{s \geq \frac{1}{2}\} - m$, where the party machine tries to win the election while expending the least possible amount of money. In addition, we could consider combinations where the party machine cares about both vote share and winning. For example, we could consider $U(s,m) = \delta s + \beta 1\{s \geq \frac{1}{2}\} - m$.

We define the Subgame Perfect Nash Equilibrium as follows,

**Definition** The strategy profile $(\lambda^*, b^*) \in \Lambda \times \mathcal{B}$ is a *Subgame Perfect Nash Equilibrium* to the request-fulfilling game if,

1. $\lambda^*(v) = \underset{\lambda(v) \in \mathbb{R}^+}{\arg\max}\, u_v(b^*, \lambda^*_{-v}, \lambda(v))$ for $v \in \mathbb{R}$
2. $b^*_\lambda = \underset{b_\lambda \in \mathcal{B}}{\arg\max}\, U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$ for all $\lambda \in \Lambda$

Throughout, integrals will take the form $\int_{v \in V} g(v) f_v(v)dv$ for some measurable subset of the real line $V$ and some measurable function $g$. We denote $\mu_v(V) = \int_{v \in V} f_v(v)dv$. We let $\bar{\mathbb{R}}^+$ denote the positive extended real line, i.e. $\bar{\mathbb{R}}^+ = \mathbb{R}^+ + \{\infty\}$. We present a number of technical lemmas that will help us characterize the results and then present the proofs of the main propositions.

### A.2 Proofs of Lemmas

This first lemma comes close to demonstrating that (ignoring sets of measure zero), the party machine will buy the cheapest voters. It demonstrates that there cannot exist two sets of equal size where the party machine buys the voters in the more expensive set, but does not buy the voters in the less expensive set.

**Lemma 1.** *Consider a history $\lambda \in \Lambda$ and a feasible party machine strategy at that history $b_\lambda \in \mathcal{B}_\lambda$. Suppose that there exist sets $V \subset \mathbb{R}$ and $V' \subset \mathbb{R}$ both with positive and equal measure $\mu_v(V) = \mu_v(V') > 0$ such that $b_\lambda(v) < \lambda(v)$ for all $v \in V$, $b_\lambda(v) > \lambda(v)$ for all $v \in V'$, and $\lambda(v) < \lambda(v')$ for all $v \in V$ and $v' \in V'$. There exists a strategy $b'_\lambda \in \mathcal{B}_\lambda$ such that,*

$$U\left(\int_v 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b'_\lambda(v) f_v(v)dv\right) \tag{5}$$

$$> U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

*Proof.* Notice that by construction $V \cap V' = \emptyset$. Define,

$$\bar{b}_{V'} = \frac{\int_{v' \in V'} b_\lambda(v') f_v(v')dv'}{\int_{v' \in V'} f_v(v')dv'} \tag{6}$$

Consider a bounded set $V'' \subset V'$ with positive measure $\mu_v(V'') > 0$. Since $\mu_v(V') > 0$, it can be shown that such a bounded set with positive measure exists. Consider the alternative strategy $b'_\lambda \in \mathcal{B}_\lambda$ defined by,

$$b'_\lambda(v) = \begin{cases} \bar{b}_{V'} - \varepsilon, & v \in V \\ 0, & v \in V' - V'' \\ \frac{\mu_v(V)}{2\mu_v(V'')}\varepsilon & v \in V'' \\ b_\lambda(v), & otherwise \end{cases} \tag{7}$$

where $\varepsilon = \bar{b}_{V'} - \sup_{v \in V} \lambda(v)$. This strategy offers to pay the "cheap" voters in $V$ slightly less than the average of what was previously paid to the expensive voters in $V'$, pays most of the expensive voters in $V'$ nothing, and uses half of the savings to keep a small portion of the expensive voters in the set $V'$.

For this definition to make sense, we have to make sure $\varepsilon = \bar{b}_{V'} - \sup_{v \in V} \lambda(v)$ is properly defined. Selecting any $v' \in V'$, we have $\sup_{v \in V} \lambda(v) < \lambda(v')$ implying that the supremum must exist. This implies that $\varepsilon$ exists. A similar argument implies that $\inf_{v \in V'} \lambda(v)$ must exist and that $\sup_{v \in V} \lambda(v) \leq \inf_{v \in V'} \lambda(v)$. Furthermore,

$$\bar{b}_{V'} = \frac{\int_{v' \in V'} b_\lambda(v') f_v(v')dv'}{\int_{v' \in V'} f_v(v')dv'} \geq \frac{\int_{v' \in V'} \lambda(v') f_v(v')dv'}{\int_{v' \in V'} f_v(v')dv'} > \inf_{v' \in V'} \lambda(v') \tag{8}$$

where the last inequality follows from the fact that $\mu_v(V') \neq 0$ and $\lambda(v') > 0$ for $v' \in V$. We therefore have,

$$\varepsilon = \bar{b}_{V'} - \sup_{v \in V} \lambda(v) > \inf_{v \in V} \lambda(v) - \sup_{v \in V} \lambda(v) \geq 0 \tag{9}$$

which guarantees that $\varepsilon > 0$.

Next, consider voters in $V$ who previously did not vote for the party machine. Under the new strategy,

$$b'_\lambda(v) - \lambda(v) = \bar{b}_{V'} - \varepsilon - \lambda(v) = \bar{b}_{V'} - \bar{b}_{V'} + \sup_{v \in V} \lambda(v) - \lambda(v) \tag{10}$$

$$= \sup_{v \in V} \lambda(v) - \lambda(v) \geq 0 \text{ for all } v \in V$$

Therefore, all voters in $V$ now vote for the party machine.

Now, consider voters in $V''$. We have,

$$b'_\lambda(v) - \lambda(v) = \tfrac{\mu_v(V)}{2\mu_v(V'')}\varepsilon - \lambda(v) \geq \tfrac{\mu_v(V)}{2\mu_v(V'')}\varepsilon - \sup_{v \in V''} \lambda(v) \text{ for all } v \in V'' \tag{11}$$

where $\sup_{v \in V''} \lambda(v)$ exists because we assumed that $V''$ was bounded. We can select the set $V''$ to have arbitrarily small measure without making $\sup_{v \in V''} \lambda(v)$ grow by selecting the smallest elements in $V'$. We therefore have that there exists a bounded set $V''$ with positive measure such that $\frac{\mu_v(V)}{2\mu_v(V'')}\varepsilon - \lambda(v) \geq 0$ for all $v \in V''$.

Now, let us consider the vote shares under the two strategies. The machine party's vote share under the new strategy is,

$$\int_{v \in \mathbb{R}} 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v) dv \tag{12}$$

$$= \int_{v \in V} 1\{(\bar{b}_{V'} - \varepsilon) \geq \lambda(v)\} f_v(v) dv + \int_{v \in V' - V''} 1\{0 \geq \lambda(v)\} f_v(v) dv$$

$$+ \int_{v \in V''} 1\{\tfrac{\mu_v(V)}{2\mu_v(V'')}\varepsilon \geq \lambda(v)\} f_v(v) dv + \int_{v \notin V \cup V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv$$

$$= \mu_v(V) + 0 + \mu_v(V'') + \int_{v \notin V \cup V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv$$

The machine party's vote share under the old strategy is,

$$\int_{v \in \mathbb{R}} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv \tag{13}$$

$$= 0 + \mu_v(V') + 0 + \int_{v \notin V \cup V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv$$

Given that $\mu_v(V) = \mu_v(V')$ by assumption and that $\mu_v(V'') > 0$ by construction, we have that,

$$\int_{v \in \mathbb{R}} 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v) dv > \int_{v \in \mathbb{R}} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv \tag{14}$$

or that the machine party's vote share increases under the new strategy.

We also compare the amount spent under each strategy. Under the new strategy,

$$\int_{v \in \mathbb{R}} b'_\lambda(v) f_v(v) dv \tag{15}$$

$$= (\bar{b}_{V'} - \varepsilon) \int_{v \in V} f_v(v)dv + 0 + \tfrac{\varepsilon \mu_v(V)}{2\mu_v(V'')} \int_{v \in V''} f_v(v)dv + \int_{v \notin V \cup V'} b_\lambda(\lambda) f_v(v)dv$$

$$= (\bar{b}_{V'} - \varepsilon)\mu_v(V) + \tfrac{\varepsilon \mu_v(V)}{2\mu_v(V'')} \mu_v(V'') + \int_{v \notin V \cup V'} b_\lambda(v) f_v(v)dv$$

$$= \bar{b}_{V'}\mu_v(V) - \tfrac{1}{2}\varepsilon \mu_v(V) + \int_{v \notin V \cup V'} b_\lambda(v) f_v(v)dv$$

Under the old strategy,

$$\int_{v \in \mathbb{R}} b_\lambda(v) f_v(v)dv \tag{16}$$

$$= \int_{v \in V} b_\lambda(v) f_v(v)dv + \int_{v \in V' - V''} b_\lambda(v) f_v(v)dv + \int_{v \in V''} 0 f_v(v)dv + \int_{v \notin V \cup V'} b_\lambda(v) f_v(v)dv$$

$$\geq \int_{v \in V} b_\lambda(v) f_v(v)dv + 0 + 0 + \int_{v \notin V \cup V'} b_\lambda(v) f_v(v)dv$$

$$= \bar{b}_{V'}\mu_v(V) + \int_{v \notin V \cup V'} b_\lambda(v) f_v(v)dv$$

Given that $-\frac{1}{2}\varepsilon\mu_v(V) < 0$, we have that,

$$\int_{v \in \mathbb{R}} b'_\lambda(v) f_v(v)dv < \int_{v \in \mathbb{R}} b_\lambda(v) f_v(v)dv \tag{17}$$

By Assumption 3, since the vote share increases and the money spent decreases under the new strategy, it follows that,

$$U\left(\int_v 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b'_\lambda(v) f_v(v)dv\right) \tag{18}$$

$$> U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

proving the result.

□

This next lemma demonstrates that the party machine will not pay voters more than it has too (again, except on a set of measure zero)—the party machine will either pay voters exactly their request, or will pay them nothing. The exception to this result occurs when the party machine has a budget large enough to buy the entire electorate. In this case, we have to additionally assume that the party machine does not waste left over money.

**Lemma 2.** *Consider a history $\lambda \in \Lambda$ and a feasible party machine strategy at that history $b_\lambda \in \mathcal{B}_\lambda$. Suppose that there exists a set $V \subset \mathbb{R}$ with positive measure $\mu_v(V) > 0$ such that $b_\lambda(v) \notin \{0, \lambda(v)\}$ for all $v \in V$. There exists a strategy $b'_\lambda \in \mathcal{B}_\lambda$ such that,*

$$U\left(\int_v 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b'_\lambda(v) f_v(v)dv\right) \tag{19}$$

$$\geq U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

*where we have a strict inequality if either* $U(1, m') > U(1, m)$ *for all* $m' > m$ *or* $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) > 0$.

*Proof.* Suppose that such a set $V$ exists. Define the subsets $V_+ \subset V$ and $V_0 \subset V$ by $V_+ = \{v \in V : b_\lambda(v) \geq \lambda(v)\}$ and $V_0 = \{v \in V : b_\lambda(v) < \lambda(v)\}$. Notice that by construction, $V_+ \cap V_0 = \emptyset$ and $V_+ \cup V_0 = V$. Consider first the case where $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) > 0$. Let $V' \subset \{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}$ be a bounded set with positive measure, $\mu_v(V') > 0$. Define,

$$\delta = \int_{v \in V_+} (b_\lambda(v) - \lambda(v)) f_v(v)dv + \int_{v \in V_0} b_\lambda(v) f_v(v)dv > 0 \tag{20}$$

Consider the following alternative strategy $b'_\lambda \in \mathcal{B}_\lambda$ defined by,

$$b'_\lambda(v) = \begin{cases} b_\lambda(v), & v \in V^c - V' \\ \lambda(v), & v \in V_+ \\ 0, & v \in V_0 - V' \\ \frac{1}{2\mu_v(V')}\delta & v \in V' \end{cases} \tag{21}$$

This strategy pays all voters in $V$ who previously voted for the party machine their minimal demand, most of the voters in $V$ who previously voted against the party machine zero, and uses half the savings to buy a small number of voters who previously voted against the party machine.

We can calculate the vote share under the new strategy to be,

$$\int_{v \in \mathbb{R}} 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv \tag{22}$$

$$= \int_{v \in V^c - V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv + \int_{v \in V_+} 1\{\lambda(v) \geq \lambda(v)\} f_v(v)dv$$

$$+ \int_{v \in V_0 - V'} 1\{0 \geq \lambda(v)\} f_v(v)dv + \int_{v \in V'} 1\{\tfrac{1}{2\mu_v(V')}\delta \geq \lambda(v)\} f_v(v)dv$$

$$= \int_{v \in V^c - V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv + \mu_v(V_+) + 0 + \int_{v \in V'} 1\{\tfrac{1}{2\mu_v(V')}\delta \geq \lambda(v)\} f_v(v)dv$$

We can select $\mu_v(V')$ to be sufficiently small so that $\frac{1}{2\mu_v(V')}\delta \geq \lambda(v)$ in which case we have,

$$\int_{v \in \mathbb{R}} 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv \tag{23}$$

$$= \int_{v \in V^c - V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv + \mu_v(V_+) + \mu_v(V')$$

We can calculate the machine party's vote share under the old strategy to be,

$$\int_{v \in \mathbb{R}} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv \tag{24}$$

$$= \int_{v \in V^c - V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv + \int_{v \in V_+} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv + \int_{v \in V_0} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv$$

$$= \int_{v \in V^c - V'} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv + \mu_v(V_+)$$

Notice that the new vote share is greater than the old vote share since $\mu_v(V') > 0$ by assumption.

Considering the new budget, we have,

$$\int_{v \in \mathbb{R}} b'_\lambda(v) f_v(v) dv \tag{25}$$

$$= \int_{v \in V^c - V'} b_\lambda(v) f_v(v) dv + \int_{v \in V_+} \lambda(v) f_v(v) dv + \int_{v \in V_0 - V'} 0 f_v(v) dv + \int_{v \in V'} \frac{1}{2\mu_v(V')} \delta f_v(v) dv$$

$$= \int_{v \in V^c - V'} b_\lambda(v) f_v(v) dv + \int_{v \in V_+} \lambda(v) f_v(v) dv + 0 + \tfrac{1}{2}\delta$$

and considering the old budget,

$$\int_{v \in \mathbb{R}} b_\lambda(v) f_v(v) dv \tag{26}$$

$$= \int_{v \in V^c - V'} b_\lambda(v) f_v(v) dv + \int_{v \in V_+} b_\lambda(v) f_v(v) dv + \int_{v \in V_0} b_\lambda(v) f_v(v) dv$$

$$= \int_{v \in V^c - V'} b_\lambda(v) f_v(v) dv + \int_{v \in V_+} \lambda(v) f_v(v) dv + 0 + \delta$$

which is greater that the new budget. It follows that under the new strategy, the party machine obtains a strictly greater vote share while spending strictly less, so Assumption 3 implies that the party machine receives strictly greater utility under $b'_\lambda$.

Consider alternatively the case where $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) = 0$. Consider the alternative strategy,

$$b''_\lambda(v) = \lambda(v) \tag{27}$$

Under this new strategy, the party machine's vote share is,

$$\int_{v \in \mathbb{R}} 1\{b''_\lambda(v) \geq \lambda(v)\} f_v(v) dv = \int_{v \in \mathbb{R}} 1\{\lambda(v) \geq \lambda(v)\} f_v(v) dv = 1 \tag{28}$$

Under the old strategy, we also have,

$$\int_{v \in \mathbb{R}} 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v) dv = 1 \tag{29}$$

Under the new strategy, the budget is equal to,

$$\int_{v \in \mathbb{R}} b''_\lambda(v) f_v(v) dv = \int_{v \in \mathbb{R}} \lambda(v) f_v(v) dv \tag{30}$$

while under the old strategy, the budget is equal to,

$$\int_{v \in \mathbb{R}} b_\lambda(v) f_v(v) dv > \int_{v \in \mathbb{R}} \lambda(v) f_v(v) dv \tag{31}$$

where the strict inequality follows from the fact that $b_\lambda(v) > \lambda(v)$ on a set of positive measure by assumption. Since the party machine achieves the same vote share of 1 while spending less, Assumption 2 implies that the utility of $b''_\lambda$ is weakly greater than the utility of $b_\lambda$. If we further assume that $U(1, m') > U(1, m)$ for all $m' > m$, we have that the utility is strictly greater.

□

The next lemma demonstrates that the party machine will use a cutoff strategy where it pays all voters exactly their request if their request is below some threshold level and will offer the voters zero otherwise. Again, we have to separately consider the case where the party machine has a very large budget.

**Lemma 3.** *Consider a history $\lambda \in \Lambda$ and a feasible party machine strategy at that history $b_\lambda \in \mathcal{B}_\lambda$. Suppose that there does not exist a $\tau \in \bar{\mathbb{R}}^+$ such that $b_\lambda(v) = \lambda(v)$ for almost all $\lambda(v) < \tau$ and $b_\lambda(v) = 0$ for almost all $\lambda(v) > \tau$. Then there exists a feasible strategy $b'_\lambda \in \mathcal{B}_\lambda$ such that,*

$$U\left(\int_v 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b'_\lambda(v) f_v(v)dv\right) \tag{32}$$

$$\geq U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

*where we have a strict inequality if either $U(1, m') > U(1, m)$ for all $m' > m$ or $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) > 0$.*

*Proof.* Suppose that there does not exists a $\tau \in \bar{\mathbb{R}}^+$ such that $b_\lambda(v) \geq \lambda(v)$ for almost all $\lambda(v) < \tau$ and $b_\lambda(v) < \lambda(v)$ for almost all $\lambda(v) > \tau$. Then there must exist sets of positive measure $V$ and $V'$ such that $b_\lambda(v) < \lambda(v)$, $b_\lambda(v') \geq \lambda(v')$, and $\lambda(v) < \lambda(v')$ for all $v \in V$ and $v' \in V'$. Without loss of generality, we can take these two set to have equal measure by taking a subset of larger set with measure equal to the smaller set. Then, by Lemma 1, there exists a $b'_\lambda \in \mathcal{B}_\lambda$ such that,

$$U\left(\int_v 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b'_\lambda(v) f_v(v)dv\right) \tag{33}$$

$$> U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

Now consider any $b_\lambda \in \mathcal{B}_\lambda$ where there is a cutoff $\tau \in \mathbb{R}^+ + \{\infty\}$ such that $b_\lambda(v) \geq \lambda(v)$ for almost all $\lambda(v) < \tau$ and $b_\lambda(v) < \lambda(x)$ for almost all $\lambda(v) > \tau$. By Lemma 2, there exists a $b''_\lambda \in \mathcal{B}_\lambda$ such that,

$$U\left(\int_v 1\{b''_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b''_\lambda(v) f_v(v)dv\right) \tag{34}$$

$$\geq U\left(\int_v 1\{b_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b_\lambda(v) f_v(v)dv\right)$$

where strict inequality holds under the conditions of the lemma.

□

The next lemma demonstrates that the party machine's strategy can be characterized by choosing the strategy that maximizes the machine utility among all cutoff strategies. We have to take some care because it is possible that there will be a positive measure of voters who request exactly the cutoff. In this case, the machine must choose not just the cutoff, but which subset of voters who request exactly the cutoff to buy. In equilibrium, the machine will buy almost all voters who request exactly the cutoff.

**Lemma 4.** *Consider a history $\lambda \in \Lambda$ and a feasible party machine strategy at that history $b_\lambda \in \mathcal{B}_\lambda$. Suppose that,*

$$b_\lambda \in \arg\max_{b'_\lambda \in \mathcal{B}} U\left(\int_v 1\{b'_\lambda(v) \geq \lambda(v)\} f_v(v)dv, B - \int_v b'_\lambda(v) f_v(v)dv\right) \tag{35}$$

*Then there exists a $\tau \in \bar{\mathbb{R}}^+$ and a measurable subset $\mathrm{T}(\lambda) \subset \{\lambda(v) = \tau\}$ satisfying,*

$$\tau \in \arg\max_{\tau' : \tau' \in \bar{\mathbb{R}}^+, \int_{\{v:\lambda(v)<\tau\}\cup \mathrm{T}(\lambda)} \lambda(v) f_v(v)dv \leq B} \tag{36}$$

$$U\left(\int_{\{v:\lambda(v)<\tau'\}\cup \mathrm{T}(\lambda)} f_v(v)dv, B - \int_{\{v:\lambda(v)<\tau'\}\cup \mathrm{T}(\lambda)} \lambda(v) f_v(v)dv\right)$$

*such that $b'_\lambda(v) = \lambda(v)$ for all $\lambda(v) < \tau$, $b'_\lambda(v) = 0$ for all $\lambda(v) > \tau$, and $U(b'_\lambda, \lambda) = U(b_\lambda, \lambda)$. If either $U(1, m') > U(1, m)$ for all $m' > m$ or $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) > 0$, then there exists a $\tau$ as defined above such that $b_\lambda(v) = \lambda(v)$ for almost all $\lambda(v) < \tau$ and $b'_\lambda(v) = 0$ for almost all $\lambda(v) > \tau$.*

*Proof.* We first demonstrate that there is a cutoff strategy with the cutoff defined by equation (36) that yields at least as much utility as $b_\lambda$. If $b_\lambda$ is not itself a cutoff strategy (except on a set of measure 0), Lemma 3 implies that there is a cutoff strategy that yields at least as much utility. Since this cutoff strategy is feasible in the optimization problem defined in equation (36), it must yield at least as much utility as $b_\lambda$.

Now, suppose further that either $U(1, m') > U(1, m)$ for all $m' > m$ or $\int_v \lambda(v) f_v(v)dv > B$. If $b_\lambda$ is a feasible strategy and $\int_v \lambda(v) f_v(v)dv > B$, it follows that $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) > 0$ since $\int_v \lambda(v) f_v(v)dv > B$ implies that the party machine's budget is not large enough to meet the requests of all the voters. We therefore have that either $U(1, m') > U(1, m)$ for all $m' > m$ or $\mu_v(\{v \in \mathbb{R} : b_\lambda(v) < \lambda(v)\}) > 0$. Applying Lemma 3 again, if $b_\lambda$ is not a cutoff strategy, there must exist a cutoff strategy that yields strictly higher utility. We therefore have that $b_\lambda$ must be a cutoff strategy. Now, suppose that the cutoff $\tau$ defining the cutoff strategy is not an optimizer of the problem defined by equation (36). It follows that there exists a different feasible cutoff strategy that yields strictly greater utility than $b_\lambda$, so $b_\lambda$ cannot solve the optimization problem defined in equation (35).

□

The final lemma uses a characterization of the machine's optimization problem as choosing a cutoff and choosing a fraction of the voters exactly at the cutoff to buy. The lemma demonstrates that this problem has a solution, even for subgames that are never reached. This step is necessary in demonstrating that an equilibrium exists. The proof works as follow. We reduce the machine's problem to a decision of how much money to spend, which is a one-dimensional optimization problem. We use the fact that a continuous function defined over a compact set attains a maximum. Special care must be taken because the machine's utility function may be left-discontinuous at $s = \frac{1}{2}$. In this special case, we break the problem up into two parts that each involve maximizing a continuous function over a compact set. There is a small region that does not fall into either of these two parts, but we show that the maximum cannot occur there because the machine utility experiences a positive jump discontinuity in that region.

**Lemma 5.** *The following is non-empty,*

$$\underset{(\tau,\kappa)\in\bar{\mathbb{R}}^+\times[0,1],\int_{v:\lambda(v)<\tau}\lambda(v)f_v(v)dv+\kappa\tau\int_{v:\lambda(v)=\tau}f_v(v)dv\leq B}{\arg\max} \tag{37}$$

$$U\left(\int_{v:\lambda(v)<\tau} f_v(v)dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v)dv, B - \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v)dv - \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v)dv\right)$$

*Proof.* The optimization problem above is equivalent to the following problem,

$$\underset{(\tau,\kappa,s,m)\in\bar{\mathbb{R}}^+\times[0,1]\times[0,1]\times[0,B]}{\arg\max} \quad U(s,m) \tag{38}$$

such that

$$s = \int_{v:\lambda(v)<\tau} f_v(v)dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v)dv \tag{39}$$

$$m = B - \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v)dv - \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v)dv \tag{40}$$

We first characterize for each $m \in [0, B]$ the $(\tau, \kappa) \in \bar{\mathbb{R}}^+ \times [0, 1]$ that solve,

$$\int_{v:\lambda(v)<\tau} \lambda(v) f_v(v)dv + \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v)dv = B - m \tag{41}$$

Consider the set,

$$\left\{\tau \in \bar{\mathbb{R}}^+ : \int_{\{v:\lambda(v)\leq\tau\}} \lambda(v) f_v(v)dv \geq B - m\right\} \tag{42}$$

Notice that $\int_{\{v:\lambda(v)\leq 0\}} \lambda(v) f_v(v)dv \geq 0$ and that,

$$\int_{\{v:\lambda(v)\leq\tau\}} \lambda(v) f_v(v)dv < \int_{\{v:\lambda(v)\leq\tau'\}} \lambda(v) f_v(v)dv \tag{43}$$

for $\tau < \tau'$. If follows that the set is bounded from below, so the infimum of the set must exist. Since the function $\int_{\{v:\lambda(v)\leq\tau\}} \lambda(v) f_v(v)dv$ is right continuous and only has jump discontinuities, the infimum must be attained at some $\tau' \in \bar{\mathbb{R}}^+$, so define $\tau' = \inf_\tau \left\{\tau \in \bar{\mathbb{R}}^+ : \int_{\{v:\lambda(v)\leq\tau\}} \lambda(v) f_v(v)dv \geq B - m\right\}$. By the definition of $\tau'$, we must have,

$$\int_{\{v:\lambda(v)\leq\tau'\}} \lambda(v) f_v(v)dv \geq B - m \tag{44}$$

$$\int_{\{v:\lambda(v)\leq\tau'-\varepsilon\}} \lambda(v) f_v(v)dv < B - m \tag{45}$$

for all $\varepsilon > 0$. The second equation implies that,

$$\int_{\{v:\lambda(v)<\tau'\}} \lambda(v) f_v(v)dv \leq B - m \tag{46}$$

If $\int_{\{v:\lambda(v)=\tau'\}} \lambda(v) f_v(v)dv = 0$, then.

$$\int_{\{v:\lambda(v)\leq\tau'\}}\lambda(v)f_v(v)dv=\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv\geq B-m \tag{47}$$

implying that,

$$\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv=\int_{\{v:\lambda(v)\leq\tau'\}}\lambda(v)f_v(v)dv=B-m \tag{48}$$

so,

$$\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv+\kappa\int_{\{v:\lambda(v)=\tau'\}}\lambda(v)f_v(v)dv \tag{49}$$
$$=\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv=B-m$$

for all $\kappa\in[0,1]$. If,

$$\int_{\{v:\lambda(v)=\tilde{\tau}\}}\lambda(v)f_v(v)dv=0 \tag{50}$$

we can select,

$$\kappa=\frac{B-m-\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv}{\int_{\{v:\lambda(v)=\tau'\}}\lambda(v)f_v(v)dv} \tag{51}$$

Notice that,

$$\int_{\{v:\lambda(v)\leq\tau'\}}\lambda(v)f_v(v)dv=\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv+\int_{\{v:\lambda(v)=\tau\}}\lambda(v)f_v(v)dv\geq B-m \tag{52}$$

which implies that,

$$B-m-\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv\leq\int_{\{v:\lambda(v)=\tau'\}}\lambda(v)f_v(v)dv \tag{53}$$

In addition,

$$\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv\leq B-m \tag{54}$$

which implies that,

$$B-m-\int_{\{v:\lambda(v)<\tau'\}}\lambda(v)f_v(v)dv\geq 0 \tag{55}$$

which together implies that,

$$\frac{B-m-\int_{\{v:\lambda(v)<\tilde{\tau}\}}\lambda(v)f_v(v)dv}{\int_{\{v:\lambda(v)=\tilde{\tau}\}}\lambda(v)f_v(v)dv}\in[0,1] \tag{56}$$

Hence, the is a unique $\tau\in\bar{\mathbb{R}}^+$ that solves,

$$\int_{v:\lambda(v)<\tau}\lambda(v)f_v(v)dv+\kappa\tau\int_{v:\lambda(v)=\tau}\lambda(v)f_v(v)dv=B-m \tag{57}$$

if $\int_{v:\lambda(v)=\tau} \lambda(v) f_v(v) dv > 0$, there is a unique $\kappa \in [0,1]$ as well, and if $\int_{v:\lambda(v)=\tau} \lambda(v) f_v(v) dv = 0$, then any $\kappa \in [0,1]$ will work.

We next show that for each $m \in [0, B]$, there exists only one $s \in [0,1]$ such that there exist $(\tau, \kappa) \in \bar{\mathbb{R}}^+ \times [0,1]$ satisfying,

$$s = \int_{v:\lambda(v)<\tau} f_v(v) dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v) dv \tag{58}$$

$$m = B - \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v) dv + \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v) dv \tag{59}$$

We know that there is a unique $\tau$ which solves the second equation. Suppose that we have $\int_{v:\lambda(v)=\tau} f_v(v) dv = 0$. Then $s = \int_{v:\lambda(v)<\tau} f_v(v) dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v) dv$ are equal for all $\kappa \in [0,1]$, so there is also a unique $s$. Suppose instead that $\int_{v:\lambda(v)=\tau} f_v(v) dv > 0$. Then there is a unique $\kappa \in [0,1]$ solving $m = B - \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v) dv + \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v) dv$ which implies that there is a unique $s \in [0,1]$ as well. Denote this unique $s \in [0,1]$ for each $m \in [0,B]$ by the map $s(m)$. We can reformulate our optimization problem as,

$$\arg\max_{m \in [0,B]} U(s(m), m) \tag{60}$$

We would like to demonstrate that $s(m)$ is strictly decreasing in $m$ for all $m \in [0, B]$. Suppose that $m < m'$. We have that,

$$\begin{aligned} &B - \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v) dv - \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v) dv \\ &< B - \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v) dv - \kappa'\tau' \int_{v:\lambda(v)=\tau'} f_v(v) dv \end{aligned} \tag{61}$$

or that,

$$\begin{aligned} &\int_{v:\lambda(v)<\tau} \lambda(v) f_v(v) dv + \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v) dv \\ &> \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v) dv + \kappa'\tau' \int_{v:\lambda(v)=\tau'} f_v(v) dv \end{aligned} \tag{62}$$

Since,

$$\int_{v:\lambda(v)\leq\tau} \lambda(v) f_v(v) dv \geq \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v) dv + \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v) dv \tag{63}$$

$$\int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v) dv + \kappa'\tau' \int_{v:\lambda(v)=\tau'} f_v(v) dv \geq \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v) dv \tag{64}$$

We have,

$$\int_{v:\lambda(v)\leq\tau} \lambda(v) f_v(v) dv > \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v) dv \tag{65}$$

which implies that $\tau > \tau'$.

Now, consider,

$$s = \int_{v:\lambda(v)<\tau} f_v(v)dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v)dv \tag{66}$$

$$s' = \int_{v:\lambda(v)<\tau'} f_v(v)dv + \kappa' \int_{v:\lambda(v)=\tau'} f_v(v)dv \tag{67}$$

We have,

$$\begin{aligned} s' = \int_{v:\lambda(v)<\tau'} f_v(v)dv + \kappa' \int_{v:\lambda(v)=\tau'} f_v(v)dv \leq \int_{v:\lambda(v)\leq\tau'} f_v(v)dv < \int_{v:\lambda(v)<\tau} f_v(v)dv \\ \leq \int_{v:\lambda(v)<\tau} f_v(v)dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v)dv = s \end{aligned} \tag{68}$$

which implies that $s' < s$ proving that $s(m)$ is strictly decreasing in $m$ for all $m \in [0, B]$.

We next would like to demonstrate that the function $s(m)$ is continuous in $m$ for all $m \in [0, B]$. We need to show that for each $\delta > 0$, there exists an $\varepsilon > 0$ such that $m' \in [0, B]$ and $|m' - m| < \varepsilon$ implies that $|s(m') - s(m)| < \delta$. We show that this is the case by selecting $\varepsilon = \tau\delta$ where $(\tau, \kappa)$ are such that,

$$m = B - \int_{v:\lambda(v)<\tau} \lambda(v) f_v(v)dv - \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v)dv \tag{69}$$

Consider an $m' < m$ with,

$$m' = B - \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v)dv - \kappa'\tau' \int_{v:\lambda(v)=\tau'} f_v(v)dv \tag{70}$$

We have that $\tau' \geq \tau$ and $s' > s$. We have,

$$|s' - s| = s' - s \tag{71}$$

$$= \int_{v:\lambda(v)<\tau'} f_v(v)dv + \kappa' \int_{v:\lambda(v)=\tau'} f_v(v)dv - \int_{v:\lambda(v)<\tau} f_v(v)dv + \kappa \int_{v:\lambda(v)=\tau} f_v(v)dv$$

$$\leq \int_{v:\lambda(v)\leq\tau'} f_v(v)dv - \int_{v:\lambda(v)<\tau} f_v(v)dv = \int_{v:\tau\leq\lambda(v)\leq\tau'} f_v(v)dv \leq \int_{v:\tau<\lambda(v)<\tau'} f_v(v)dv$$

$$\leq \tfrac{1}{\tau} \int_{v:\tau<\lambda(v)<\tau'} \lambda(v) f_v(v)dv \leq \tfrac{1}{\tau} \left[ \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v)dv - \int_{v:\lambda(v)\leq\tau} \lambda(v) f_v(v)dv \right]$$

$$\leq \tfrac{1}{\tau} \left[ \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v)dv + \kappa'\tau' \int_{v:\lambda(v)=\tau'} f_v(v)dv - \int_{v:\lambda(v)\leq\tau} \lambda(v) f_v(v)dv - \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v)dv \right]$$

$$= \tfrac{1}{\tau} \left( B - \int_{v:\lambda(v)\leq\tau} \lambda(v) f_v(v)dv - \kappa\tau \int_{v:\lambda(v)=\tau} f_v(v)dv \right)$$

$$-\tfrac{1}{\tau} \left( B - \int_{v:\lambda(v)<\tau'} \lambda(v) f_v(v)dv - \kappa'\tau' \int_{v:\lambda(v)=\tau'} f_v(v)dv \right)$$

$$= \tfrac{1}{\tau}(m - m') = \tfrac{1}{\tau}|m' - m| < \varepsilon = \tfrac{1}{\tau}(\tau\delta) = \delta$$

We thus have that $s(m)$ is continuous in $m$ for all $m \in [0, B]$.

Now define,

$$s(0) = \underline{s} \tag{72}$$

$$s(B) = \bar{s} \tag{73}$$

$$\tilde{U}(m) = U(s(m), m) \tag{74}$$

Suppose that $\frac{1}{2} \notin [\underline{s}, \bar{s}]$. Since $s(m)$ is strictly decreasing in $m$, there does not exist an $m \in [0, B]$ such that $s(m) = \frac{1}{2}$. In that case,

$$\arg\max_{m\in[0,B]} U(s(m), m) = \arg\max_{m\in[0,B]} \tilde{U}(m) \tag{75}$$

is non-empty because it involves maximizing a continuous function over a compact set. Alternatively, suppose that $\underline{s} \leq \frac{1}{2} \leq \bar{s}$. Then there exists an $m$ such that $s(m) = \frac{1}{2}$ and we have that $\tilde{U}(m)$ is continuous at all $m$ except $m = s^{-1}(\frac{1}{2}) \in [0, B]$. We also have that $\tilde{U}(m)$ is right continuous at $s^{-1}(\frac{1}{2})$ and experiences a positive jump discontinuity there. We can write,

$$\arg\max_{m\in[0,B]} U(s(m), m) = \arg\max_{m\in[0,B]} \tilde{U}(m) \tag{76}$$

$$= \arg\max\{ \arg\max_{m\in[0,s^{-1}(\frac{1}{2})]} \tilde{U}(m), \arg\max_{m\in[s^{-1}(\frac{1}{2}),B]} \tilde{U}(m)\}$$

We have that $\arg\max_{m\in[s^{-1}(\frac{1}{2}),B]} \tilde{U}(m)$ is clearly nonempty because it involves the maximization of a continuous function over a compact set. We must only show that $\arg\max_{m\in[0,s^{-1}(\frac{1}{2})]} \tilde{U}(m)$ is non-empty. Suppose that there exists an $\varepsilon > 0$ such that,

$$\arg\max_{m\in[0,s^{-1}(\frac{1}{2})]} \tilde{U}(m) = \arg\max_{m\in[0,s^{-1}(\frac{1}{2})-\varepsilon]} \tilde{U}(m) \tag{77}$$

Then the maximum exists. If there does not exists a $\varepsilon > 0$ such that,

$$\arg\max_{m\in[0,s^{-1}(\frac{1}{2})]} \tilde{U}(m) = \arg\max_{m\in[0,s^{-1}(\frac{1}{2})-\varepsilon]} \tilde{U}(m) \tag{78}$$

it must be the case that,

$$\sup_{m\in[0,s^{-1}(\frac{1}{2})]} \tilde{U}(m) \leq \lim_{\varepsilon\downarrow 0} \max_{m\in[0,s^{-1}(\frac{1}{2})-\varepsilon]} \tilde{U}(m) \leq \tilde{U}(s^{-1}(\tfrac{1}{2})) \tag{79}$$

so the maximum exists, proving the result.

□

### A.3 Proofs of Main Results

**Proposition 4.** *Consider the equilibrium strategy profile $(\lambda^*, b^*) \in \Lambda \times \mathcal{B}$. There exists a $\tau^* \in \mathbb{R}^+$ such that,*

1. *$\lambda^*(v) = \tau^*$ for all $v \in (-\infty, \tau^*]$ except on a set of measure zero and $\lambda^*(v) > \tau^*$ for all $v \in [\tau, \infty)$ except on a set of measure zero.*
2. *$b^*_{\lambda^*}(v) = \tau^*$ for all $v \in (-\infty, \tau^*]$ except on a set of measure zero and $b^*_{\lambda^*}(v) = 0$ for all $v \in [\tau, \infty)$ except on a set of measure zero.*
3. *Voters with $v \in (-\infty, \tau^*]$ vote for the party machine, except on a set of measure zero, and voters with $v \in (\tau^*, \infty)$ vote against the party machine, except on a set of measure zero.*
4. *$\tau^* \in \underset{\tau:\tau\geq 0, \tau F_v(\tau)\leq B}{\arg\max} U(F_v(\tau), B - \tau F_v(\tau))$*

*Proof.* Suppose first that there exists a $\tau$ with $f_v(\tau) > 0$ such that $\tau F_v(\tau) > B$. Let $\bar{v} = \sup(\{v \in \mathbb{R} : f_v(v) > 0\})$ where we allow $\bar{v} = \infty$. Then $\tau \leq \bar{v}$. Consider an equilibrium where the party machine buys all voters, i.e.,

$$\int_{v\in\mathbb{R}} 1\{b^*_{\lambda^*}(v) \geq \lambda^*(v)\} f_v(v) dv = 1 \tag{80}$$

Since the equilibrium strategy must be feasible, we must have,

$$\int_{v\in\mathbb{R}} \lambda^*(v) f_v(v) dv \leq \int_{v\in\mathbb{R}} b^*_{\lambda^*}(v) f_v(v) dv \leq B \tag{81}$$

We have,

$$\int_{v\in\mathbb{R}} \tau f_v(v) dv \geq \int_{v\leq\tau} \tau f_v(v) dv = \tau F_v(\tau) > B \geq \int_{v\in\mathbb{R}} \lambda^*(v) f_v(v) dv \tag{82}$$

which implies that $\lambda^*(v) < \tau$ for all $v \in \mathbb{R}$. Now, consider a voter with bias $v$. This voter gets utility $-v + \lambda^*(v)$ under the proposed equilibrium. Suppose that $b^*_{\lambda^*}$ is such that the party would have honored a larger request by the voter. Then the proposed equilibrium cannot be an equilibrium. Hence, it must be the case that this voter could receive a utility of 0 by making a larger request (which won't be fulfilled by the party machine). In order for $\lambda^*(v)$ to be optimal for such a voter, it must be the case that $-v + \lambda^*(v) > 0$, or that $v < \lambda^*(v) < \tau$. Since $\tau < \bar{v}$ by assumption, there must be a positive mass of voters with $\tau < v < \bar{v}$ who are not playing equilibrium strategies. Hence, we cannot have an equilibrium where the party machine buys every voter if there exists a $\tau$ with $f_v(\tau) > 0$ such that $\tau F_v(\tau) > B$. We thus have that Assumption 5 implies the conditions for obtaining the result that a cutoff strategy is strictly optimal from Lemma 4.

Applying Lemma 4 at history $\lambda^*$, there exists a $\tau \in \mathbb{R}^+$ such that $b_{\lambda^*}(v) = \lambda^*(v)$ for almost all $\lambda^*(v) < \tau$ and $b_{\lambda^*}(v) = 0$ for almost all $\lambda^*(v) > \tau$. Consider a voter with bias $v \leq \tau$. For almost all such voters, the voter gets utility $-v + \lambda^*(v)$ if it makes a request $\lambda^*(v) \leq \tau$ and receives utility 0 if it makes a higher request. If it makes the request $\lambda^*(v) = \tau$, it receives utility $-v + \tau \geq 0$. If it makes request $\lambda^*(v) < \tau$, it will receive $-v + \lambda^*(v) < -v + \tau$, so almost all such voters can maximize their utility by making the requesting $\lambda^*(v) = \tau$ and their offers will be accepted, indicating they will vote for the party machine. Consider alternatively a voter with $v > \tau$. Almost all such voters receive $-v + \lambda^*(v)$ by making the request $\lambda^*(v) \leq \tau$ and receive 0 otherwise. We have $-v + \lambda^*(v) < -\tau + \lambda^*(v) \leq 0$, so almost all voters with $v > \tau$ will prefer to make a request that is not accepted. Consequently, almost all such voters will request $\lambda^*(v) > \tau$ and will vote against the party machine.

Applying what we found about $\lambda^*$ here, we have from Lemma 4 that,

$$\tau^* \in \underset{\tau':\tau'\in\bar{\mathbb{R}}^+,\int_{\{v:\lambda^*(v)<\tau\}\cup\mathrm{T}(\lambda^*)}\lambda^*(v)f_v(v)dv\le B}{\arg\max} \tag{83}$$

$$U\left(\int_{\{v:\lambda^*(v)<\tau\}\cup\mathrm{T}(\lambda^*)} f_v(v)dv, B - \int_{\{v:\lambda^*(v)<\tau\}\cup\mathrm{T}(\lambda^*)} \lambda^*(v)f_v(v)dv\right)$$

$$= \underset{\tau':\tau'\ge 0,\int_{v\le\tau'}\tau' f_v(v)dv\le B}{\arg\max} U\left(\int_{v\le\tau'} f_v(v)dv, B - \int_{v\le\tau'} \tau' f_v(v)dv\right)$$

$$= \underset{\tau':\tau'\ge 0,\tau' F_v(\tau')\le B}{\arg\max} U\left(F_v(\tau'), B - \tau' F_v(\tau')\right)$$

proving the result.

□

**Proposition 5.** *There exists an equilibrium to the vote-buying game.*

*Proof.* By Lemma 5,

$$\underset{(\tau,\kappa)\in\bar{\mathbb{R}}^+\times[0,1],\int_{v:\lambda(v)<\tau}\lambda(v)f_v(v)dv+\kappa\tau\int_{v:\lambda(v)=\tau}f_v(v)dv\le B}{\arg\max} \tag{84}$$

$$U\left(\int_{v:\lambda(v)<\tau} f_v(v)dv + \kappa\int_{v:\lambda(v)=\tau} f_v(v)dv, B - \int_{v:\lambda(v)<\tau}\lambda(v)f_v(v)dv - \kappa\tau\int_{v:\lambda(v)=\tau} f_v(v)dv\right)$$

is non-empty for each history $\lambda \in \Lambda$. Let $(\tau(\lambda), \kappa(\lambda))$ denote such a solution for each history and let $(\tau^*, \kappa^*) = (\tau(\lambda^*), \kappa(\lambda^*))$. Let $\mathrm{T}(\lambda)$ be a measurable subset of $\{v \in \mathbb{R} : \lambda(v) = \tau(\lambda)\}$ with measure $\mu_v(\mathrm{T}(\lambda)) = \kappa\mu_{\mathrm{v}}(\{\mathrm{v} \in \mathbb{R} : \lambda(\mathrm{v}) = \tau(\lambda)\})$. We use $\tau(\lambda)$ and $\mathrm{T}(\lambda)$ to construct an equilibrium. We define,

$$b^*_\lambda(v) = \begin{cases} \lambda(v), & \lambda(v) < \tau(\lambda) \text{ or } v \in \mathrm{T}(\lambda) \\ 0, & \textit{otherwise} \end{cases} \tag{85}$$

$$\lambda^*(v) = \begin{cases} \tau^*, & v \le \tau^* \\ \infty, & v > \tau^* \end{cases} \tag{86}$$

To show that there is no profitable deviation for the party machine at any history $\lambda$, consider a $b_\lambda \ne b^*_\lambda$. We first consider cutoff strategies. By construction, no cutoff strategy can yield weakly higher utility at any of the subgames. Lemma 4 demonstrates that no other strategy can yield weakly higher utility than the optimal cutoff strategy. Hence, there is no profitable deviation for the party machine.

The second part in the proof involves showing that there is no profitable deviation for any voter. Consider a voter with $v \le \tau^*$. The voter receives $-v + \lambda(v)$ if $\lambda(v) \le \tau^*$ and 0 otherwise. If the voter deviates to a lower demand $\lambda(v) \le \tau^*$, we receives $-v + \lambda(v) < -v + \tau^*$. If the voter deviates to a higher demand, we receives 0. For this to be a profitable deviation, we must have $0 > -v + \tau^*$ for $v \le \tau^*$ which cannot be the case. Now, considering a voter with $v > \tau^*$, the voter receives 0. Only deviations with $\lambda(v) \le \tau$ change his utility. Such a deviation gives utility $-v + \lambda(v) < -v + \tau^* < 0$. Hence, voters with $v > \tau^*$ cannot improve their utility proving that this constitutes an equilibrium.

□

## A.4 Proof for Competitive Clientelism

*Proof of Proposition 3.* Because subgame perfect equilibrium is our solution concept, we can solve this game by backwards induction. We first consider the strategy that the parties play when observing the voters play the strategy $\lambda$. For each voter $x$, party 1 will offer either zero or will choose $b_1(x)$ to satisfy $b_1(x) - b_2(x) = \lambda(x)$. That is, $b_1(x) \in \{0, \lambda(x) + b_2(x)\}$. Party 2 will offer either zero or will chose $b_2(x)$ to satisfy $b_1(x) - b_2(x) = \lambda(x)$. That is, $b_2(x) \in \{0, -\lambda(x) + b_1(x)\}$. If a party chooses to buy a voter, it will do so as cheaply as possible, and that is accomplished by paying the voter exactly enough—$\lambda(x)+b_2(x)$ for party 1 and $-\lambda(x)+b_1(x)$ for party 2. If a party pays less than these amounts, the party can save money by paying the voter nothing.

Next, party 1 will choose to buy the cheapest voters. The cheapest voters are those with the lowest value of $\lambda(x) + b_2(x)$. We thus has that there is some cutoff $\tau_1 \geq 0$ such the party pays exactly $\lambda(x) + b_2(x)$ for all voters with $\lambda(x) + b_2(x) \leq \tau_1$ and will pay zero to all other voters. Similarly, party 2 will pay all voters $-\lambda(x) + b_1(x)$ as long as $-\lambda(x) + b_1(x) \leq \tau_2$.

At most one party will pay a positive payment if the voters vote with probability 1 for one of the two parties. Now, consider a positive mass of voters who vote for both parties with positive probability. For these voters, we must have $b_1(x) - b_2(x) = \lambda(x)$. Each party can pay these voters $\varepsilon$ more and increase the probability that the voter votes for the party to 1. The party can redistribute some of its budget from an arbitrarily small measure of voters and give an arbitrary small amount to the voters who mix between the parties and increase its' vote share. Therefore, the "tied" voters must have measure zero and can therefore be ignored.

So to summarize,

- Voters with $\lambda(x) + b_2(x) \leq \tau_1$ will receive $b_1(x) = \lambda(x) + b_2(x)$ and $b_2(x) = 0$
- Voters with $-\lambda(x) + b_1(x) \leq \tau_2$ will receive $b_1(x) = 0$ and $b_2(x) = -\lambda(x) + b_1(x)$
- All other voters receive nothing.

or more succinctly,

- Voters with $\lambda(x) \leq \tau_1$ will receive offers $b_1(x) = \lambda(x)$ and $b_2(x) = 0$
- Voters with $\lambda(x) \geq -\tau_2$ will receive offers $b_1(x) = 0$ and $b_2(x) = -\lambda(x)$
- All other voters receive nothing.

Now, we move to the first stage. The voter realizes that if she requests $\lambda(x) = \tau_1$, she will vote for party 1 and receive utility $-\gamma x + \tau_1$, if she requests $\lambda(x) = -\tau_2$, she will receive $-\gamma(1-x) + \tau_2$, if she requests $\lambda(x) > \tau_1$, she will receive $-\gamma(1-x)$, if she requests $\lambda(x) < -\tau_2$, she will receive $-\gamma x$, if she requests $0 \leq \lambda(x) < \tau_1$, she will receive $-\gamma x + \lambda(x)$, and if she requests $-\tau_2 < \lambda(x) \leq 0$, she will receive $-\gamma(1-x) + \lambda(x)$. Clearly, the voters will not consider the last two possibilities. Furthermore, the voters will not request possibilities 3 and 4 since the voters can still vote for this party and receive a positive benefit. Hence, we have $\lambda(x) \in \{\tau_1, -\tau_2\}$ for almost all voters.

Consider the $x$ satisfying $-\gamma x + \tau_1 = -\gamma(1-x) + \tau_2$. We have $x = \frac{\tau_1 - \tau_2 + \gamma}{2\gamma}$. All voters with $x \leq \frac{\tau_1 - \tau_2 + \gamma}{2\gamma}$ request $\lambda(x) = \tau_1$ and all voters with $x \geq \frac{\tau_1 - \tau_2 + \gamma}{2\gamma}$ request $\lambda(x) = -\tau_2$. In order for the parties budgets to be balanced, we must have,

$$\tau_1 \frac{\tau_1 - \tau_2 + \gamma}{2\gamma} = B_1 \tag{87}$$

$$\tau_2(1 - \frac{\tau_1 - \tau_2 + \gamma}{2\gamma}) = B_2 \tag{88}$$

Define $d = \frac{\tau_1 - \tau_2 + \gamma}{2\gamma}$. We can write,

$$\tau_1 d = B_1 \tag{89}$$

$$\tau_2(1 - d) = B_2 \tag{90}$$

$$(\tau_1 + \gamma - 2\gamma d)(1 - d) = B_2 \tag{91}$$

Provided that $d \neq 0$, we have,

$$\tau_1 = B_1 d^{-1} \tag{92}$$

$$2d^3 - 3d^2 + (1 - \frac{B_1}{\gamma} - \frac{B_2}{\gamma})d + \frac{B_1}{\gamma} = 0 \tag{93}$$

The second equation is cubic, so it has three complex roots, but we can show that there is only one real root between zero and one. An equilibrium must have $d \in (0, 1)$ since all voters lie on the unit interval.

Now, consider the function $f(d) = 2d^3 - 3d^2 + (1 - \frac{B_1}{\gamma} - \frac{B_2}{\gamma})d + \frac{B_1}{\gamma}$. We have $f(-\infty) = -\infty < 0$, $f(0) = \frac{B_1}{\gamma} > 0$, $f(1) = -\frac{B_2}{\gamma} < 0$, and $f(\infty) = \infty > 0$. By the continuous mapping theorem, $f(d)$ must have a root between $-\infty$ and 0, between 0 and 1, and between 1 and $\infty$. Since a cubic function can have at most 3 roots, there must be exactly one root between 0 and 1, proving that there exists a unique $d$ defining the equilibrium. Because $\tau_1 = B_1 d^{-1}$ and $d = \frac{\tau_1 - \tau_2 + \gamma}{2\gamma}$, there must also be unique $\tau_1$ and $\tau_2$ defining the equilibrium. It thus follows that there is a unique $x^*$ defining the equilibrium, where voters with $x \leq x^*$ vote for party 1 and receive benefit $\tau_1$ and voters with $x > x^*$ vote for party 2 and receive benefit $\tau_2$, and where $\tau_1$ and $\tau_2$ are the unique solutions to (87) and (88).

□