Online Appendix B

In this appendix, we demonstrate that the result of Proposition 1 goes through under much weaker conditions than Assumption 4. In particular, we no longer require the individual traits to cover the entire latent space. Instead, we allow the individual traits to take on as few as 3 distinct values. We drop the assumption that $\mathcal{N} = \mathbb{N}$ and instead require $\mathcal{N} \subset \mathbb{N}$. We replace Assumption 4 with,

Assumption 5. *For any* $\varepsilon > 0$ *and* $x \in \mathbb{R}^{D+1}$*, there exists a* $t \in \mathbb{N}$ *such that* $||(a_{t0}, \boldsymbol{b}_{t0}) - \boldsymbol{x}|| < \varepsilon$*. In addition, there exists an* $n* \in \mathcal{N}$ *such that* $\theta_{n*10} \notin \{-1, 0, 1\}$.

Notice that the conditions on $\boldsymbol{\theta}_{n0}$ are extremely weak. In addition to the requirement that $\boldsymbol{\theta}_{n0}$ span $\mathbb{R}^D$, we assume that there exists a dimension with an individual whose individual trait lies between two other individuals on that dimension. In the unidimensional case, this only requires that we observe 3 distinct individual traits. In the multidimensional case, Assumption 3 and Assumption 5 imply that we need to observe at least $D + 2$ distinct individual traits.

The proof of Proposition 4 is more difficult than the proof of Proposition 1. In particular, the argument makes use of Renyi's (1957) β-expansion theorem. The proof employing the more restrictive Assumption 4 is given in the body of the paper because this proof is easier to follow and more likely to translate to related problems where an argument resembling the one given below would be much more difficult to apply.

Proposition 4. *Suppose that Assumptions 1-3 and 5 hold. Then the item response model is nonparametrically identified in the homogeneous case.*

Proof. The proof is identical up until Equation (25), replacing $n \in \mathbb{N}$ with $n \in \mathcal{N}$ throughout. We apply $y = (y_1, 0, ..., 0)$ and $n = n^*$ to (25) to obtain

$$m(x) + [m(x + y_1) - m(x)]m(\theta_{n*10}) = m(x + y_1\theta_{n*10}) \text{ for all } (x, y_1) \in \mathbb{R}^2. \tag{B1}$$

From (21), we have

$$m(0) = 0. \tag{B2}$$

Using the fact that $m(1) = F^{-1}(F_0(1)) = F^{-1}(p) = 1$, we have

$$m(1) = 1. \tag{B3}$$

Using $x = 0$ in (B1), we have

$$m(y_1\theta_{n*10}) = m(y_1)m(\theta_{n*10}) \text{ for all } y_1 \in \mathbb{R}. \tag{B4}$$

Using $y_1 = 1$ in (B1), we have

$$m(x + \theta_{n*10}) = m(x) + [m(x + 1) - m(x)]m(\theta_{n*10}) \text{ for all } x \in \mathbb{R}. \tag{B5}$$

Using $x = \theta_{n*10}$ in (B1), we have $m(1+y_1) = 1+[m(\theta_{n*10}+y_1)-m(\theta_{n*10})]$, or, equivalently,

$$m(x + 1) = m(x + \theta_{n*10}) - m(\theta_{n*10}) + 1 \text{ for all } x \in \mathbb{R}. \tag{B6}$$

We plug (B6) into (B5) to obtain

$$[m(\theta_{n*10} + x) - m(x) - m(\theta_{n*10})][m(\theta_{n*10}) - 1] = 0 \text{ for all } x \in \mathbb{R}. \tag{B7}$$

If $m(\theta_{n*10}) = 1$, we have then (B6) implies that $m(x+1) = m(x+\theta_{n*10})$ which cannot hold for all $x \in \mathbb{R}$ if $\theta_{n*10} \neq 1$. Hence, we must have

$$m(\theta_{n*10}) \neq 1. \tag{B8}$$

Using (B8) in (B7), we have

$$m(\theta_{n*10} + x) = m(\theta_{n*10}) + m(x) \text{ for all } x \in \mathbb{R}. \tag{B9}$$

Next, if $m(\theta_{n*10}) = 0$, then (B9) implies that $m(\theta_{n*10} + x) = m(x)$ which cannot hold for all $x \in \mathbb{R}$ if $\theta_{n*10} \neq 0$. Hence, we must have,

$$m(\theta_{n*10}) \neq 0. \tag{B10}$$

Next, Lemma 1 implies that m is continuously differentiable. We take derivatives of both sides of (B4) with respect to y_1 to obtain

$$m'(\theta_{n*10} y_1) = \tfrac{m(\theta_{n*10})}{\theta_{n*10}} m'(y_1) \text{ for all } y_1 \in \mathbb{R}. \tag{B11}$$

We can take derivatives of both sides of (B9) with respect to x to obtain

$$m'(x + \theta_{n*10}) = m'(x) \text{ for all } x \in \mathbb{R}. \tag{B12}$$

Consider any integer $k \geq 1$. We can iteratively apply (B11),

$$m'(\theta_{n*10}^k y_1) = \tfrac{m(\theta_{n*10})}{\theta_{n*10}} m'(\theta_{n*10}^{k-1} y_1) = \left[\tfrac{m(\theta_{n*10})}{\theta_{n*10}}\right]^2 m'(\theta_{n*10}^{k-2} y_1) = ...$$

$$= \left[\tfrac{m(\theta_{n*10})}{\theta_{n*10}}\right]^k m'(y_1), \tag{B13}$$

to obtain

$$m'(\theta_{n*10}^k y_1) = \left[\tfrac{m(\theta_{n*10})}{\theta_{n*10}}\right]^k m'(y_1) \text{ for all } k \in \mathbb{N} \text{ and } y_1 \in \mathbb{R}. \tag{B14}$$

Equation (B14) holds for $k = 0$ trivially. We can rearrange (B11) to obtain

$$m'(y_1) = \tfrac{\theta_{n*10}}{m(\theta_{n*10})} m'(\theta_{n*10} y_1) \text{ for all } y_1 \in \mathbb{R}. \tag{B15}$$

Letting $z = \theta_{n*10} y_1$, we have $m'(\theta_{n*10}^{-1} z) = \tfrac{\theta_{n*10}}{m(\theta_{n*10})} m'(z)$, or, equivalently,

$$m'(\theta_{n*10}^{-1} y_1) = \tfrac{\theta_{n*10}}{m(\theta_{n*10})} m'(y_1) \text{ for all } y_1 \in \mathbb{R}. \tag{B16}$$

We can apply (B16) iteratively to yield

$$m'(\theta_{n*10}^{-k} y_1) = \left[\tfrac{m(\theta_{n*10})}{\theta_{n*10}}\right]^{-k} m'(y_1) \text{ for all } k \in \mathbb{N} \text{ and } y_1 \in \mathbb{R}. \tag{B17}$$

Combining these results, we have

$$m'(\theta_{n*10}^k y_1) = \left[\frac{m(\theta_{n*10})}{\theta_{n*10}}\right]^k m'(y_1) \text{ for all } k \in \mathbb{Z} \text{ and } y_1 \in \mathbb{R}. \tag{B18}$$

Next, for any $k \in \mathbb{Z}$, we have

$$m'(x + \theta_{n*10}^k) = m'(x\theta_{n*10}^{k-1}\theta_{n*10}^{-k+1} + \theta_{n*10}^{k-1}\theta_{n*10}) = m'(\theta_{n*10}^{k-1}[x\theta_{n*10}^{-k+1} + \theta_{n*10}])$$

$$\text{for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}. \tag{B19}$$

We can apply (B18) to (B19) to obtain

$$m'(x + \theta_{n*10}^k) = \left[\frac{m(\theta_{n*10})}{\theta_{n*10}}\right]^{k-1} m'(x\theta_{n*10}^{-k+1} + \theta_{n*10}) \text{ for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}. \tag{B20}$$

We can apply (B12) to (B20) to obtain

$$m'(x + \theta_{n*10}^k) = \left[\frac{m(\theta_{n*10})}{\theta_{n*10}}\right]^{k-1} m'(\theta_{n*10}^{-k+1} x) \text{ for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}. \tag{B21}$$

Applying (B19) to (B21), we have

$$m'(x + \theta_{n*10}^k) = m'(x) \text{ for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}. \tag{B22}$$

Consider any integer $j \geq 1$. We can iteratively apply (B22) to obtain

$$m'(x+j\theta_{n*10}^k)=m'(x+(j-1)\theta_{n*10}^k)=...=m'(x) \text{ for all } j\in\mathbb{N},\ k\in\mathbb{Z}, \text{ and } x\in\mathbb{R}. \quad \text{(B23)}$$

Similarly, we can obtain

$$m'(x+j\theta_{n*10}^k)=m'(x) \text{ for all } j\in\mathbb{Z},\ k\in\mathbb{Z}, \text{ and } x\in\mathbb{R}. \quad \text{(B24)}$$

Consider any positive integer $r\geq 1$ and any sequence of integers $(j_0,...,j_r)$. We can apply (B24) recursively to obtain

$$m'(x+\sum_{k=0}^{r} j_k\theta_{n*10}^k)=m'(x) \text{ for all } r\in\mathbb{N},\ (j_0,...,j_r)\in\mathbb{Z}^{r+1}, \text{ and } x\in\mathbb{R}. \quad \text{(B25)}$$

Suppose that $\theta_{n*10}>1$. We can define $\beta=\theta_{n*10}$. Renyi's (1957) β-expansion theorem implies that for any $\beta>1$ and $z\in\mathbb{R}$, there exists an infinite series representation, $z=\sum_{i=0}^{\infty}c_i\beta^i$, where c_i are integers. Since the series is convergent, for all $\varepsilon>0$, there exists an $r\in\mathbb{N}$ such that $|z-\sum_{i=0}^{r}c_i\beta^i|<\varepsilon$. Hence, the theorem implies that for all $z\in\mathbb{R}$ and $\varepsilon>0$, there exists an $r\in\mathbb{N}$ and a $(j_0,...,j_r)\in\mathbb{Z}^{r+1}$ such that $\left|\sum_{k=0}^{r}j_k\theta_{n*10}^k-z\right|<\varepsilon$. Since m' is continuous, for all $\delta>0$, there exists an $\varepsilon>0$ such that $\left|\sum_{k=0}^{r}j_k\theta_{n*10}^k-z\right|<\varepsilon$ implies that $\left|m'(x+\sum_{k=0}^{r}j_k\theta_{n*10}^k)-m'(x+z)\right|<\delta$. Since this holds for all $\delta>0$, it must be the case that

$$m'(x+z)=m'(x) \text{ for all } (x,z)\in\mathbb{R}^2. \quad \text{(B26)}$$

Similarly, if $0 < \theta_{n*10} < 1$, we can define $\beta = \theta_{n*10}^{-1}$ and obtain the same result. The result applies if $\theta_{n*10} < 0$ and $\theta_{n*10} \neq -1$ since the integers j_k are allowed to be negative in (B25). Hence, (B26) holds when $\theta_{n*10} \notin \{-1, 0, 1\}$. Now, setting $y = x + z$ in (B26), we have

$$m'(y) = m'(x) \text{ for all } (x, y) \in \mathbb{R}^2. \tag{B27}$$

The result implies that m is linear. This combined with (B2) and (B3) implies that

$$m(x) = x \text{ for all } x \in \mathbb{R}. \tag{B28}$$

To complete the proof, continue from Equation (28) in the proof of Proposition 1.

□

References

Renyi, A. (1957). Representations for Real Numbers and their Ergodic Properties. *Acta Mathematica Hungarica, 8,* 477–493.