Note Set 9 – Dynamic Optimization Problems and Dynamic Games

9.1 – Overview

Often, we will be interested in solution dynamic optimization problems, where an agent maximizing the discounted stream of utilities $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$ subject to some constraints on the stream x_t. In this note set, we will cover some of the theory behind these problems as well as some numerical approaches for solving these problems. We will see that we can reformulate these problems in "recursive form" by introducing value functions. We can then solve these problems by applying fixed point iterations to a finite dimensional approximation to the functional equation that defines the value function.

9.2 – Contraction Mappings

Consider a map T of the space X into itself. We say that $x*$ is a fixed point of T if we have $T(x*) = x*$. Often, we would like to find the fixed point of a mapping T. Notice that in the case where $X = \mathbb{R}^D$, finding a fixed point is much like solving a nonlinear system. For example, we can transform any non-linear system $f(x) = 0$ into a fixed point problem by defining $T(x) = x - f(x)$. Here, we have $f(x*) = 0$ if an only if $T(x*) = x*$. Conversely, we can transform a fixed-point problem into a non-linear system by specifying $f(x) = T(x) - x$.

We say that T is a contraction mapping if $\|T(x) - T(y)\| \leq \beta \|x - y\|$ for some $\beta < 1$ and all $x, y \in X$. Finding a fixed point of contraction mappings is much easier than the general problem of solving a non-linear system of equations. A few results will explain this. First, a contraction mapping is guaranteed to have a unique solution- i.e. both existence and uniqueness are guaranteed. Furthermore, the fixed point can be count by applying fixed point iterations to T.

Before we present the theory behind these results, we need to introduce the idea of a metric space. This allow us to introduce the theory in a level of generality that the applies to both finite and functional problems.

Definition: A *Metric Space* is a set S and a distance function $\rho : S \times S \to \mathbb{R}$ such that for all $s, t, u \in S$, we have,

(i) $\rho(s,t) \geq 0$ and $\rho(s,t) = 0$ only if $s = t$.

(ii) $\rho(s,t) = \rho(t,s)$

(iii) $\rho(s,u) \le \rho(s,t) + \rho(t,u)$

We say that a metric space (S, ρ) is complete if every Cauchy sequence in S converges to an element in S. A sequence $\{s_n\}_{n=1}^{\infty}$ in S is a Cauchy sequence if for each $\varepsilon > 0$, there exists an N_ε such that $\rho(s_m, s_n) < \varepsilon$ for all $m, n \ge N_\varepsilon$.

<u>Theorem 9.1 (Contraction Mapping Theorem)</u>: If (S, ρ) is a complete metric space and $T: S \to S$ is a contraction mapping with modulus $\beta < 1$, then,

(i) T has exactly one fixed point $s^* \in S$.

(ii) For any $s_0 \in S$, $\rho(T^n s_0, s^*) \le \beta^n \rho(s_0, s^*)$ for $n = 0,1,2,...$

<u>Proof</u>: (i) We will prove that a fixed point exists by finding such a point. Define $s_n = T^n s_0$ and let s^* be the limit of the sequence $\{s_n\}_{n=1}^{\infty}$. To prove that this series converges, we must show that for each $\varepsilon > 0$, there exists an N_ε such that $\rho(s_m, s_n) < \varepsilon$ for all $m, n \ge N_\varepsilon$. Without loss of generality, we will show that this holds for all $m > n$. Since T is a contraction mapping, we have,

$$\rho(s_{n+1}, s_n) = \rho(Ts_n, Ts_{n-1}) \le \beta\rho(s_n, s_{n-1}) = \beta\rho(Ts_{n-1}, Ts_{n-2}) \le \beta^2 \rho(s_{n-1}, s_{n-2})$$

$$\le \beta^3 \rho(s_{n-2}, s_{n-3}) \le ... \le \beta^n \rho(s_1, s_0)$$

By the triangular inequality, we have,

$$\rho(s_m, s_n) \le \rho(s_m, s_{m-1}) + \rho(s_{m-1}, s_{m-2}) + ... + \rho(s_{n+1}, s_n)$$

$$\le \beta^{m-1}\rho(s_1, s_0) + \beta^{m-2}\rho(s_1, s_0) + ... + \beta^n \rho(s_1, s_0)$$

$$= \beta^n(\beta^{m-n-1} + \beta^{m-n-2} + ... + 1)\rho(s_1, s_0) \le \frac{\beta^n}{1-\beta}\rho(s_1, s_0)$$

This implies that the sequence converges to some point $s*$. Since S is complete, it must be the case that $s* \in S$. The final step it to show that $Ts* = s*$. Notice that,

$$\rho(Ts*, s*) \le \rho(Ts*, T^n s_0) + \rho(T^n s_0, s*) \le \beta\rho(s*, T^{n-1}s_0) + \rho(T^n s_0, s*)$$
$$\rho(T^n s_0, s*) = \rho(TT^{n-1}s_0, s*)$$

for all n. Now, we already know that each of these terms converges to zero as $n \to \infty$. Hence, we have that $\rho(Ts*, s*) = 0$, or $Ts* = s*$. Finally, we must show that no other point $s \in S$ satisfies $Ts = s$. Consider any point $Ts = s$. Notice that,

$$\rho(s, s*) = \rho(Ts, Ts*) \le \beta\rho(s, s*)$$

which implies that,

$$\rho(s, s*)(1-\beta) \le 0$$

Since $\beta < 1$, this cannot hold unless $\rho(s, s*) = 0$, proving that $s = s*$. Hence, the fixed point is unique.

(ii) Notice that,

$$\rho(T^n s, s*) = \rho(T^n s, Ts*) \le \beta\rho(T^{n-1}s, s*) \le \beta^2\rho(T^{n-2}s, s*) \le ... \le \beta^n\rho(s_0, s*)$$

which suffices to prove the result.

The following shortcut is useful in proving that a mapping is a contraction mapping,

Theorem 9.2 (Blackwell's Theorem): Let $X \subseteq \mathbb{R}^J$ and let $B(X)$ be the space of bounded function $f : X \to \mathbb{R}$, with the sup-norm. Let $T : B(X) \to B(X)$ be an operator satisfying,

(i) (Monotonicity) If $f, g \in B(X)$ and $f(x) \le g(x)$ for all $x \in X$ implies that $(Tf)(x) \le (Tg)(x)$, for all $x \in X$.

(ii) (Discounting) There exists a $\beta \in (0,1)$ such that $[T(f+a)](x) \le (Tf)(x) + \beta a$ for all $f \in B(X)$, $a \ge 0$, and $x \in X$.

When a non-linear system can be written as a contraction mapping, we will always want to use this approach into order to solve for the root of the non-linear system (incidentally, applying the contraction mapping theorem is one of the useful approaches to prove existence and uniqueness). In the next few sections, we will learn in what situations we are likely to be able to apply contraction mappings. We will show how to apply fixed point iterations even when the underlying mapping is not a contraction mapping.

8.3 – Dynamic Programming

Consider an unemployed individual who receives a job offer in each period. Job offers are drawn from the distribution F_s which has support on $[0, \bar{s}]$. An individual may choose to accept or reject the job offer in each period. If he accepts the job offer, he receives that salary for the rest of his life. Otherwise, he receives nothing in that period. Let x_t denote the individuals income in period t. Then the individuals utility function is given by, $\sum_{t=0}^{\infty} \beta^t x_t$.

Now, let $a_t(h_t) = 1$ if the individual accepts the job offer in period t and $a_t(h_t) = 0$. Here $h_t = (s_0, s_1, ..., s_t)$ denotes the history of salary offers. The individuals optimization problem can be specified as,

$$\max_{(a_0, a_1, ...)} E_0 \left[\sum_{t=1}^{\infty} \beta^t x_t \right]$$

such that $x_t = \begin{cases} s_{t^*}, & t \geq t^* \\ 0, & otherwise \end{cases}$ where t^* is the smallest t such that $a_t = 1$

This problem is specified in sequence form, where the individuals strategy in each period is a contingency plan.

In order to apply the theory of contraction mapping, we rewrite the above problem in recursive form. We let $v(s)$ denote the “value” of receiving an offer of s. Given this offer, the individual can receive $s + \beta s + \beta^2 s + ... = \frac{1}{1-\beta} s$. Alternatively, he can receive $0 + \beta E_{s'}[v(s')]$. Hence, we can write,

$$v(s) = \max \left\{ \tfrac{1}{1-\beta} s, \beta E_{s'}[v(s')] \right\}$$

This is a functional equation in v. We can write this as $v = Tv$. When we write the problem this way, it is say to be in recursive form.

We can look for a solution of the form,

$$v(s) = \begin{cases} \frac{1}{1-\beta} s, & s \geq s^* \\ v^*, & otherwise \end{cases}$$

Notice first that $E_{s'}[v(s')] = F_s(s^*)v^* + (1 - F_s(s^*)) * E[s \mid s \geq s^*]$. We require that,

$$v^* = F_s(s^*)v^* + (1 - F_s(s^*)) * E[s \mid s \geq s^*]$$

$$\tfrac{1}{1-\beta} s^* = \beta v^*$$

Combining these equations, we have,

$$\tfrac{1}{\beta(1-\beta)} s^* = E[s \mid s \geq s^*]$$

In the special case where $s \sim U[0,\overline{s}]$, we have,

$$s^* = \frac{\beta(1-\beta)}{2-\beta(1-\beta)}\overline{s}$$

More generally, we can solve for the equilibrium as a single nonlinear equations, but let us consider solving this as a functional fixed point problem (we may have to compute $E[s \mid s \geq s^*]$ using numerical integration).

Let us start by showing that T is a contraction mapping. Consider any u, v, and let use select ρ to be the sup-norm. We have,

$$\rho(Tu,Tv) = \sup_{0 \leq s \leq \overline{s}} \left| \max\left\{ \tfrac{1}{1-\beta} s, \beta E_{s'}[u(s')] \right\} - \max\left\{ \tfrac{1}{1-\beta} s, \beta E_{s'}[v(s')] \right\} \right|$$

Suppose, without loss of generality, that $E_{s'}[u(s')] < E_{s'}[v(s')]$. We have,

$$\rho(Tu,Tv) = \max\left\{ \sup_{s<(1-\beta)\beta E_{s'}[u(s')]} \left| \beta E_{s'}[u(s')] - \beta E_{s'}[v(s')] \right|, \right.$$

$$\left. \sup_{(1-\beta)\beta E_{s'}[u(s')] \leq s \leq (1-\beta)\beta E_{s'}[v(s')]} \left| \beta E_{s'}[u(s')] - \tfrac{1}{1-\beta} s \right| \right\}$$

$$\leq \beta(E_{s'}[v(s') - u(s')]) \leq \beta(E_{s'} \mid v(s') - u(s') \mid) \leq \beta \sup_{0 \leq s \leq \overline{s}} \left| u(s) - v(s) \right| = \beta\rho(u,v)$$

The results show that T is a contraction mapping, and hence, has a unique fixed point. Hence, one effective algorithm we can use is to discretize the space and apply fixed point iterations.

Now, to solve the problem, we can approximate the functional equation using a finite system of equations. We let $\tilde{s}_j$ for $j = 1, 2, ..., J$ denote a grid of points on $[0, \bar{s}]$. We have,

$$v_i = \max\left\{ \tfrac{1}{1-\beta} \tilde{s}_j, \beta \sum_{k=1}^{J} \tilde{p}_k v_k \right\}$$

Notice that we must approximate the distribution F_s using finite approximation, $\tilde{p}_j$. A reasonable approach is to use $\tilde{p}_j = F_s\left(\frac{\tilde{s}_j + \tilde{s}_{j+1}}{2}\right) - F_s\left(\frac{\tilde{s}_{j-1} + \tilde{s}_j}{2}\right)$ for $1 < j < J$, $\tilde{p}_1 = F_s\left(\frac{\tilde{s}_0 + \tilde{s}_1}{2}\right)$, and $\tilde{p}_J = 1 - F_s\left(\frac{\tilde{s}_{J-1} + \tilde{s}_J}{2}\right)$.

9.4 – Dynamic Campaign Spending by Political Parties

Consider a long-lived political party that receives resources $d > 0$ in each period. In each period, the party can spend some fraction x other their resources. If the party spends x, its probability of winning the election is $\gamma_1 - (\gamma_1 - \gamma_0)e^{-x}$ where $1 > \gamma_1 > \gamma_0 > 0$. If the party does not spend its resources, it earns an interest rate of $r > 0$. We assume that the period utility of the party depends on the probability of winning the election, but that utility is discounted at a rate of $0 < \beta < 1$.

We can write the problem in the following form,

$$\max_{x_1, x_2, \ldots} \sum_{t=0}^{\infty} \beta^t \left\{ \gamma_1 - (\gamma_1 - \gamma_0)e^{-x_t} \right\} \text{ such that } 0 \le x_t \le y_t + d,\ y_{t+1} = (1+r)(y_t + d - x_t)$$

Let us first reformulate this problem in recursive form,

$$v(y) = \max_{0 \le x \le y+d} \left\{ \gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta v((1+r)(y + d - x)) \right\}$$

Proposition 8.3: The Map,

$$(Tv)(y) = \max_{0\le x\le y+d} \left\{\gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta v((1+r)(y+d-x))\right\}$$

is a contraction mapping.

Proof: We prove this result by invoking Blackwell's Theorem. We must first show that $w(y) \ge v(y)$ for all y implies that $(Tw)(y) \ge (Tv)(y)$ for all y. Suppose that,

$$x^* \in \arg\max_{0\le x\le y+d} \left\{\gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta v((1+r)(y+d-x))\right\}$$

Then we have,

$$\max_{0\le x\le y+d} \left\{\gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta w((1+r)(y+d-x))\right\}$$

$$\ge \gamma_1 - (\gamma_1 - \gamma_0)e^{-x^*} + \beta w((1+r)(y+d-x^*))$$

$$\ge \gamma_1 - (\gamma_1 - \gamma_0)e^{-x^*} + \beta v((1+r)(y+d-x^*))$$

$$= \max_{0\le x\le y+d} \left\{\gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta v((1+r)(y+d-x))\right\}$$

proving the monotonicity condition.

The second condition requires that $[T(v+a)](y) \le [T(v)](y) + \beta a$. We have,

$$[T(v+a)](x) = \max_{0\le x\le y+d} \left\{\gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta(v((1+r)(y+d-x)) + a)\right\}$$

$$= \max_{0\le x\le y+d} \left\{\gamma_1 - (\gamma_1 - \gamma_0)e^{-x} + \beta v((1+r)(y+d-x))\right\} + \beta a$$

$$= [T(v)](x) + \beta a \le [T(v)](x) + \beta a$$

proving the result.

We will solve the model by applying fixed-point iterations. Let us try the parameterization, $d = 1$, $\beta = 0.9$, $r = 1.05$, $\gamma_1 = 0.6$, $\gamma_2 = 0.4$. Let us use a grid of $M = 201$ points on $[0,10]$. Using this approach, we obtained the results below. Notice that the strategy the party follows is to spend the initial endowment fairly quickly, and then continue to the entire endowment of $d = 1$ is each future period.

Figure 9.1 – Value Function

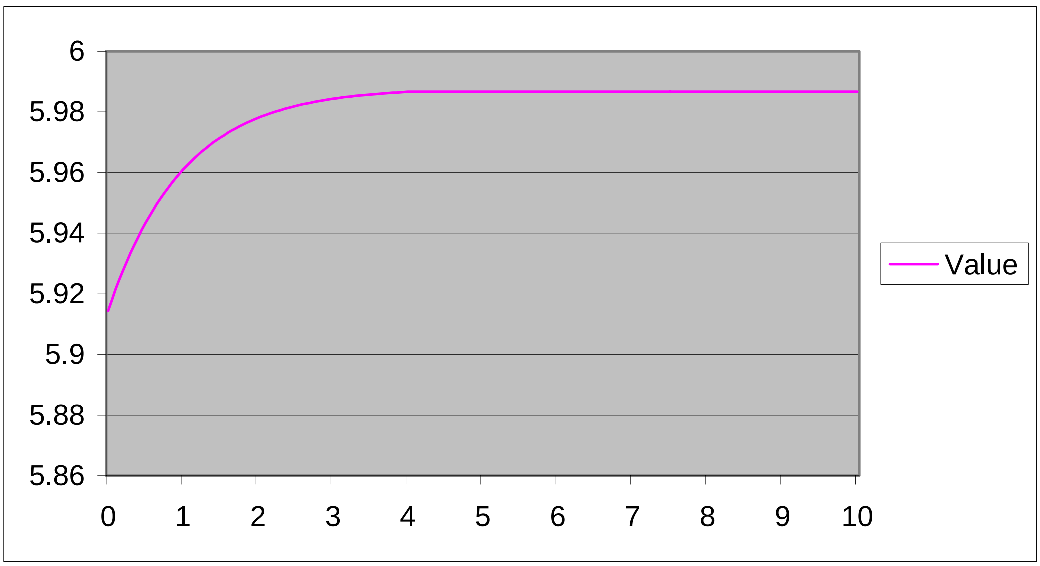
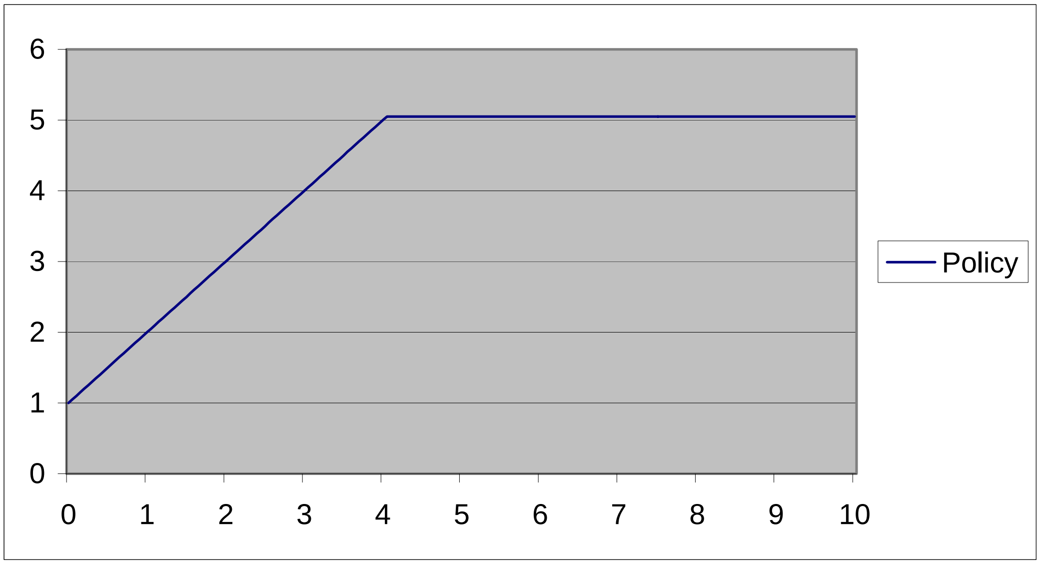


Figure 9.2 – Policy Function



9.5 – The Random Proposer Model

We consider the following model. The legislature consists of N members. Policy outcomes are characterized by $x \in X$ where X is the policy space. In each period, a random member of the legislature is selected to propose a bill b. Legislator n is selected with probability p_n. Each member of the legislature can vote for or against the proposal. Each legislator has voting weight w_n. A proposal requires as least q votes to pass. If the sum of the weights of legislators voting for the proposal is greater than or equal to q, the bill passes and the game ends. Otherwise, the game continues. Utilities are discounted at constant rate β.

We will look for a stationary equilibrium where an individual's proposal strategy $b^*(n)$ only depends on his index (and not the past history). We look an equilibrium where an individual's voting strategy $v_n^*(b) \in \{0,1\}$ depends only on the proposed bill b. Here, $v_n^*(b) = 1$ if the individual votes for the proposed bill. We define $v^*(b) = 1\left\{\sum_{n=1}^{N} w_n v_n^*(b) \geq q\right\}$. We let $W_n(b)$ denote the value to legislator n of voting over the proposal b. We assume that an individual votes for the bill if the bill gives him at least as much utility as the continuation value. We can specify.

$$W_n(b) = \max_{v\in\{0,1\}} \left\{ \begin{aligned} &1\left\{vw_n + \sum_{m\neq n} v_n^*(b)\} \geq q\right\} U_n(b) \\ &+1\left\{vw_n + \sum_{m\neq n} v_n^*(b)\} < q\right\} \beta \sum_{m=1}^{N} p_m W_n(b^*(m)) \end{aligned} \right\}$$

We also need to consider the proposals incentive. The proposer will simply choose b to satisfy,

$$b^*(n) = \arg\max_b \left\{v^*(b)U_n(b) + (1 - v^*(b))W_n(b)\right\}$$

We have the following system of equations,

(1) $$v_n^*(b) = \begin{cases} 1, & U_n(b) \geq \beta \sum_{m=1}^{N} p_m W_n(b^*(m)) \\ 0, & otherwise \end{cases}$$

(2) $$v^*(b) = 1\left\{\sum_{n=1}^{N} w_n v_n^*(b) \geq q\right\}$$

(3) $$b^*(n) = \arg\max_b \left\{v^*(b)U_n(b) + (1 - v^*(b))W_n(b)\right\}$$

(4) $$W_n(b) = \max_{v\in\{0,1\}} \left\{ \begin{array}{l} 1\left\{ vw_n + \sum_{m\neq n} v_n*(b)\} \geq q \right\} U_n(b) \\ +1\left\{ vw_n + \sum_{m\neq n} v_n*(b)\} < q \right\} \beta \sum_{m=1}^{N} p_m W_n(b*(m)) \end{array} \right\}$$

The recursive structure of some of these equations means that we can reduce the problem. In particular, let $W = W_1,...,W_N$ and let $b* = (b*(1),...,b*(N))$. Let $\upsilon = (W, b*)$. The we can write equations (3) and (4) as a fixed point, $\upsilon = T\upsilon$, where the definition of $v*$ and v_n* is given by (1) and (2). We can then apply fixed point iterations to characterize the solution.

The next step involves discretizing the problem. We will approximate X with a finite grid of points, $\tilde{x}_1,...,\tilde{x}_J$. We can write the four above equations in discrete form as,

(1') $$U_{n,j} = U_n(\tilde{x}_j)$$

(2') $$\tilde{W}_{n,m} = W_{n,j} \text{ where } b*(n) = \tilde{x}_j$$

(3') $$v_{n,j}* = \begin{cases} 1, & U_{n,j} \geq \beta \sum_{m=1}^{N} p_m W_{n,m} \\ 0, & otherwise \end{cases}$$

(4') $$v_j* = 1\left\{ \sum_{n=1}^{N} w_n v_{n,j}* \geq q \right\}$$

(5') $$b_n* = \tilde{x}_j \text{ where } j = \arg\max_{1\leq j\leq J} \left\{ v_j*U_{n,j} + (1-v_j*)W_{n,j} \right\}$$

(6') $$W_{n,j} = \max_{v\in\{0,1\}} \left\{ \begin{array}{l} 1\left\{ vw_n + \sum_{m\neq n} v_{n,j}*\} \geq q \right\} U_{n,j} \\ +1\left\{ vw_n + \sum_{m\neq n} v_{n,j}*\} < q \right\} \beta \sum_{m=1}^{N} p_m \tilde{W}_{n,m} \end{array} \right\}$$

Notice that (1) can be pre-computed. Set (2) requires $O(N^2)$ operations. Computing (3) requires $O(N^2 J)$ operations, while (4) requires $O(N)$ operations. Overall, the process requires $O(N^2 J)$ operations per iteration.

A More Compact Representation of the Random Proposer Model

Kalandrakis (2006) show that we can represent the problem in a more compact form. We assume that in equilibrium, voters will approve a proposal if it yield them a reservation level of utility, r_n. We can write the problem as,

(1) $$r_n = \beta \sum_{m=1}^{N} p_m U_n(b^*(n;r))$$

(2) $$b^*(n;r) = \underset{b:\sum_{m=1}^{N} w_m 1\{U_m(b) \geq r_m\} \geq q}{\arg\max} U_n(b)$$

We can combine these two equations to get a single one,

$$r_n = \beta \sum_{m=1}^{N} p_m U_m \left(\underset{b:\left(\sum_{m=1}^{N} w_m 1\{U_m(b) \geq r_m\}\right) \geq q}{\arg\max} U_n(b) \right)$$

Which is a fixed point equation in the reservation values.

Notice that to iterate the fixed point equation, we must solve the sub-problem,

$$\arg\max_b U_n(b) \text{ such that } \left(\sum_{n=1}^{N} w_n 1\{U_n(b) \geq r_n\} \geq q \right)$$

This is a constrained maximization problem. In principle, we could solve this problem using an NLP algorithm. In practice, this is unlikely to be effective. The set,

$\left\{ b : \left(\sum_{n=1}^{N} w_n 1\{U_n(b) \geq r_n\} \geq q \right) \right\}$ is unlikely to be convex, and may not even be connected.

To solve this problem effectively, we will have to search over a grid of points. If we solve the inner maximization problem over a grid of J, the overall computational cost is $O(N^2 J)$ per iteration, so the overall cost of this approach is the same as the previous approach.

A More Realistic Model of Legislative Bargaining

Let us consider a more complicated and more realistic model of legislative bargaining. In this case, a bill $b_0 \in X$ is introduced. The status quo is $s \in X$. In each period, a random member of the legislator is selection (with recognition probability p_n). This member may either move for the previous question $q = 1$ or propose an amendment $a \in X$. If the amendment passes, it replaces the current bill. If the individual proposes to order the previous question, the legislature votes on whether to continue debate. If the legislature votes to end debate, then a final voting round is held between the amended bill and the status quo.

We let $W_n^{\infty}(b)$ denote the value to player n of reaching the last period with b as the bill. We have,

$$W_n^{\infty}(b) = 1\left\{ \sum_{m=1}^{N} w_m 1\{U_m(b) \geq U_m(s)\} \geq q \right\} U_n(b) + 1\left\{ \sum_{m=1}^{N} w_m 1\{U_m(b) \geq U_m(s)\} < q \right\} U_n(s)$$

This expression simply indicates that the voters vote sincerely in the final period. Next, we let $V_n(b, a)$ denote the value to player n of voting over amendment a where b is the current for of the bill. We let $W_n(b)$ denote the value of individual n of voting to end debate when the current for of the bill is b. We let $a_n*(b)$ and $q_n*(b)$ denote the policy functions of individual n, which indicate whether the individual orders the previous question, and if not, what amendment he proposes. We have,

$$V_n(b,a) = \max_{v\in\{0,1\}}\left\{\begin{array}{l} 1\{\sum_{m\neq n} w_m v_m*(b,a) + w_n v \geq q\}\beta\sum_{m=1}^{N} p_m\left[q*(m)V_n(a,a*(m)) + (1-q*(m))W_n(a)\right] \\ +1\{\sum_{m\neq n} w_m v_m*(b,a) + w_n v < q\}\beta\sum_{m=1}^{N} p_m\left[q*(m)V_n(b,a*(m)) + (1-q*(m))W_n(b)\right] \end{array}\right\}$$

$$W_n(b) = \max_{v\in\{0,1\}}\left\{\begin{array}{l} 1\{\sum_{m\neq n} w_m y_m*(b) + w_n v \geq q\}\beta W_n^{\infty}(b) \\ +1\{\sum_{m\neq n} w_m y_m*(b) + w_n v < q\}\beta\sum_{m=1}^{N} p_m\left[q*(m)V_n(b,a*(m)) + (1-q*(m))W_n(b)\right] \end{array}\right\}$$

Finally, we define the policy functions as,

$$a_n*(b) = \arg\max_a W_n(b,a), \qquad q_n*(b) = \begin{cases} 0, & W_n(b,a_n*(b)) \geq V_n(b) \\ 1, & otherwise \end{cases}$$

The vote function are given by,

$$v_n*(b,a) = 1\left\{\sum_{m=1}^{N} p_m V_n(a,a_n*(a)) \geq \sum_{m=1}^{N} p_m V_n(b,a_n*(b))\right\}$$

$$y_n*(b) = 1\left\{W_n^{\infty}(b) \geq \sum_{m=1}^{N} p_m V_n(b,a*(b))\right\}$$

In order to set up the fixed point correspondence, we need to iterate (V_n, W_n, a_n*, q_n*).

9.6- Suggested Reading

[1] Baron, David, and John Ferejohn (1989). “Bargaining in legislatures”. *American Political Science Review* 83:1181–1206.

[2] Stokey, Nancy L. and Robert E. Lucas (1989). Recursive Methods in Economic Dynamics.

[3] Kalandrakis, Tasos (2006). “Regularity of Pure Strategy Equilibrium Points in a Class of Bargaining Games”. *Economic Theory* 28:309-329.

[4] Duggan, John, and Tasos Kalandrakis (2007). “Dynamic Legislative Policy Making”. Working Paper.

[5] Penn, Elizabeth Maggie (2007). “A Model of Farsighted Voting”. Working Paper.